

CONVERGENCE OF A NUMERICAL SCHEME FOR A COUPLED SCHRÖDINGER–KDV SYSTEM

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ABSTRACT. We prove the convergence in a strong norm of a finite difference semi-discrete scheme approximating a coupled Schrödinger–KdV system on a bounded domain. This system models the interaction of short and long waves. Since the energy estimates available in the continuous case do not carry over to the discrete setting, we rely on a suitably truncated problem which we prove reduces to the original one. We present some numerical examples to illustrate our convergence result. Nonlinear Schrödinger equation and Korteweg–de Vries equation and short wave long wave interaction and finite difference scheme

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1. INTRODUCTION

In [2], D.J. Benney presents a general theory modeling the nonlinear interaction between short waves and long waves, deriving nonlinear differential systems describing these interactions in various physical settings. The (complex-valued) short waves $u(x, t)$, $x \in \mathbb{R}$, $t \geq 0$, are described by a nonlinear Schrödinger equation and the (real-valued) long waves $v(x, t)$ satisfy a quasilinear equation, eventually with a dispersive term. In the most general context, the interaction is described by the nonlinear system

$$\begin{cases} i\partial_t u + ic_1\partial_x u + \partial_{xx}u = \alpha v u + \beta|u|^2 u \\ \partial_t v + c_2\partial_x v + \mu\partial_x^3 v + \nu\partial_x v^2 = \gamma\partial_x(|u|^2), \end{cases}$$

where $c_1, c_2, \alpha, \beta, \gamma, \mu$ and ν are real constants.

In this paper, we are concerned with the numerical approximation of the solutions to the Cauchy problem for the system comprising the nonlinear Schrödinger equation coupled with a Korteweg–de Vries equation with Dirichlet boundary conditions on a bounded domain $(0, L)$,

$$(1.1a) \quad i\partial_t u + \partial_{xx}u = \alpha v u + \beta|u|^2 u$$

$$(1.1b) \quad \partial_t v + \partial_x^3 v + \partial_x(v^2) = \gamma\partial_x|u|^2$$

$$(1.1c) \quad u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad x \in (0, L), \quad L > 0,$$

$$(1.1d) \quad u(0, t) = u(L, t) = 0, \quad v(0, t) = v(L, t) = 0, \quad t \in [0, T], \quad T > 0.$$

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This kind of system arises in several fields of physics such as the study of resonant interactions, short and long capillary-gravity waves on water [8], an electron plasma interaction [11] and a diatomic lattice system [13].

The Cauchy problem for the system (1.1) was initially studied on the whole line by M. Tsutsumi [12], who proved that for initial data (u_0, v_0) in $H^{m+1/2}(\mathbb{R}) \times H^m(\mathbb{R})$, $m = 1, 2, \dots$, the problem is well-posed in the same space. After that, Bekiranov *et al.* [1] established local well-posedness for initial data in $H^s(\mathbb{R}) \times H^{s-1/2}(\mathbb{R})$, $s \geq 0$, and more recently, Corcho and Linares [4] proved global well-posedness in the energy space $H^1(\mathbb{R}) \times H^1(\mathbb{R})$.

It is worth pointing out that in the above well-posedness results, uniqueness is obtained only in some subspace of $C([0, T]; H^s(\mathbb{R}) \times H^{s-1/2}(\mathbb{R}))$ (in [1]), (resp. a subspace of $C([0, T]; H^1(\mathbb{R}) \times H^1(\mathbb{R}))$ in [4]). Additionally, the techniques used in the papers [1, 4] (which were introduced by Bourgain [3] and developed by Kenig, Ponce and Vega [9, 10]), do not seem to be applicable to the generalized KdV equation (gKdV). In a very recent paper, Dias *et al.* [5], using energy methods, obtain a global solution in $(H^1(\mathbb{R}))^2$ for a coupled Schrödinger–gKdV system.

In this paper, we prove a convergence result for a semi-discrete finite difference approximation of the system (1.1) in the space $(H^1(0, L))^2$. The energy methods used by M. Tsutsumi [12] to prove global existence of a solution fail in the discrete setting, so we propose a new approach: by an appropriate truncation of the quadratic function v^2 appearing in the equation (1.1b), we consider a related problem involving a gKdV equation. For each fixed value of the truncation parameter, we are able to prove the convergence of a numerical scheme toward the solutions of this auxiliary problem. These solutions, in turn, satisfy an energy inequality. Lastly, using this energy inequality, we are able to derive an L^∞ estimate independent of the truncation parameter, which implies that the truncated problem in fact reduces to the original one.

In contrast to previous work, the proof of these stability estimates require that we work on a bounded subset $(0, L) \subset \mathbb{R}$. From the viewpoint of the applications and numerical approximation, this presents no great loss in generality.

Note also that our convergence proof does not rely on any previous existence results, and so constitutes a new existence proof for the Cauchy problem (1.1). Additionally, the present work represents, to the authors' best knowledge, the first numerical treatment of the system (1.1)–(1.1c).

An outline of the paper follows. After some notations and preliminaries, we state in Section 2 the main convergence result, Theorem 2.1. In Section 3, we prove Theorem 2.1 and present Proposition 3.2, our main auxiliary result, dealing with the convergence of approximate solutions to a suitably truncated system, and an energy estimate. Its proof is the object of Section 4. Finally, in the last section of the paper we illustrate our convergence result with some numerical simulations and check its accuracy by testing it against some known exact solutions.

1.1. Notations and preliminaries.

Let us introduce the Banach spaces

$$X_{J,\mathbb{C}} = \{z^h = (z_0, z_1, \dots, z_{J+1}) \in \mathbb{C}^{J+2} : z_0 = z_1 = z_J = z_{J+1} = 0\}$$

with $J \in \mathbb{N}_0$ and $h = L/(J+1)$. In a similar way, we define the real space $X_{J,\mathbb{R}}$. When no ambiguity arises, we represent either of these spaces by X_J . The scalar

product is given by

$$(z^h, w^h) = \sum_{j=2}^{J-1} h z_j \bar{w}_j, \quad z^h, w^h \in X_J,$$

and the p -norms by

$$\|z^h\|_{p,h} = \left(\sum_{j=2}^{J-1} h |z_j|^p \right)^{1/p}, \quad 1 \leq p < \infty; \quad \|z^h\|_\infty = \max_{j=2,\dots,J-1} |z_j|, \quad z^h \in X_J.$$

To simplify the notation we write $\|z\|_p$ for the norm of z in both the continuous and the discrete case. We denote by $H^m(0, L)$, $H_0^m(0, L)$ and $H^{-m}(0, L)$ ($m \in \mathbb{N}$) the usual Sobolev spaces. All the norms appearing in this paper are in $(0, L)$, so for instance $\|u\|_2$ means $\|u\|_{L^2(0,L)}$.

We will use the following notations for the finite difference operators. For $z = (z_j)$,

$$\begin{aligned} D_+ z_j &= \frac{z_{j+1} - z_j}{h}, & D_- z_j &= \frac{z_j - z_{j-1}}{h}, \\ D_0 z_j &= \frac{z_{j+1} - z_{j-1}}{2h} = \frac{1}{2}(D_+ + D_-)z_j, \\ \Delta^h z_j &= D_- D_+ z_j = D_+ D_- z_j = \frac{z_{j+1} - 2z_j + z_{j-1}}{h^2}, \\ D^3 z_j &= D_0 D_- D_+ z_j = \frac{z_{j+2} - 2z_{j+1} + 2z_{j-1} - z_{j-2}}{2h^3}. \end{aligned}$$

For $u = (u_j)$, let us now introduce the piecewise linear interpolator,

$$(1.2) \quad \mathbf{P}_1^h u(x) = u_j + (x - x_j) \frac{u_{j+1} - u_j}{x_{j+1} - x_j}, \quad x \in (x_j, x_{j+1}),$$

and the piecewise constant interpolator,

$$\mathbf{P}_0^h u(x) = u_j, \quad x \in (x_j, x_{j+1}).$$

Let $u^h \in X_J$. From (1.2) we derive

$$(1.3) \quad \|\mathbf{P}_1^h u^h - \mathbf{P}_0^h u^h\|_2 \leq Ch \|D_+ u^h\|_2$$

for some C independent of h . As a consequence, we obtain

$$(1.4) \quad \|\mathbf{P}_1^h u^h\|_2 \leq \|\mathbf{P}_1^h u^h - \mathbf{P}_0^h u^h\|_2 + \|\mathbf{P}_0^h u^h\|_2 \leq C \|u^h\|_2.$$

The following lemma establishes some inequalities which will be of use throughout.

Lemma 1.1. *Let $\phi = (\phi_j) \in X_J$. Then*

$$(1.5) \quad \|\phi\|_\infty \leq \sqrt{2} \|\phi\|_2^{1/2} \|D_\pm \phi\|_2^{1/2},$$

$$(1.6) \quad \|\phi\|_\infty \leq \frac{1}{2} \|D_\pm \phi\|_1 \leq \frac{\sqrt{L}}{2} \|D_\pm \phi\|_2$$

$$(1.7) \quad \|\phi\|_2 \leq C \|\phi\|_\infty,$$

where C is a constant independent of h .

Proof. The inequality (1.5) is a consequence of the Gagliardo–Nirenberg inequality and $\partial_x \mathbf{P}_1^h = \mathbf{P}_0^h D_+$:

$$\|\phi\|_\infty = \|\mathbf{P}_1^h \phi\|_\infty \leq \sqrt{2} \|\mathbf{P}_1^h \phi\|_2^{1/2} \|\partial_x \mathbf{P}_1^h \phi\|_2^{1/2} \leq \sqrt{2} \|\phi\|_2^{1/2} \|D_\pm \phi\|_2^{1/2},$$

while (1.6) is a consequence of the (continuous) inequality $\|\phi\|_\infty \leq \frac{1}{2} \|\phi'\|_1$. The last inequality is an elementary consequence of the definition of the discrete norms. \square

2. STATEMENT OF THE MAIN RESULT

We propose the following semidiscrete finite difference approximation to the Cauchy problem (1.1):

$$(2.1a) \quad i\partial_t u^h + \Delta^h u^h = \beta |u^h|^2 u^h + \alpha v^h u^h$$

$$(2.1b) \quad \partial_t v^h + D^3 v^h + D_0 (v^h)^2 = \gamma D_0 |u^h|^2$$

$$(2.1c) \quad u^h(0) = u_0^h, \quad v^h(0) = v_0^h,$$

$$(2.1d) \quad u^h(t), v^h(t) \in X_J, \quad t \in [0, T].$$

The global existence proof of Tsutsumi [12] relies on energy methods which we cannot carry over to the finite difference framework. It turns out that in the semidiscrete case, the crux of our convergence argument relies on an *a priori* L^∞ bound. However, this bound is only available for a modified problem (see Proposition 3.2 below). To deal with this difficulty, we use the fact that, under the right conditions, this problem reduces to the original one.

Our main result establishes the convergence of the approximations (2.1) toward a global weak solution of the problem (1.1) in the space $(H^1(0, L))^2$.

Theorem 2.1. *Let α, β, γ be such that $\alpha\gamma > 0$, $(u_0, v_0) \in (H_0^1(0, L))^2$ and let $(u^h, v^h) \in (C([0, T]; X_J))^2$ be the global solutions of the discretized problem (2.1) with initial data (u_0^h, v_0^h) , such that $\mathbf{P}_1^h u_0^h \rightarrow u_0$ and $\mathbf{P}_1^h v_0^h \rightarrow v_0$ in $H^1(0, L)$ as $h \rightarrow 0$. Then, up to a subsequence,*

$$\begin{aligned} \mathbf{P}_1^h u^h &\overset{*}{\rightharpoonup} u, & \mathbf{P}_1^h v^h &\overset{*}{\rightharpoonup} v & \text{in } L^\infty([0, T]; H^1(0, L)), \\ \mathbf{P}_1^h u^h &\rightarrow u, & \mathbf{P}_1^h v^h &\rightarrow v & \text{in } L^\infty([0, T]; L^2(0, L)), \end{aligned}$$

with

$$\begin{aligned} (u, v) &\in (L^\infty([0, T]; H_0^1(0, L)))^2 \\ &\cap (C([0, T]; H_0^1(0, L)) \times C([0, T]; L^2(0, L))), \quad T > 0, \end{aligned}$$

a weak solution of the Schrödinger–KdV system (1.1).

3. PROOF OF THEOREM 2.1 AND STABILITY ESTIMATES

In this section we prove Theorem 2.1 along with the necessary stability estimates. We begin with the definition of an appropriate truncated problem. For each $M > 1$ we define C^∞ functions f^M, g^M satisfying

$$f^M(v) = \begin{cases} v^2, & \text{if } |v| \leq M, \\ |v|, & \text{if } |v| > M^2 + 1, \end{cases}$$

and

$$g^M(v) = \begin{cases} v, & \text{if } |v| \leq M, \\ \pm C, & \text{if } |v| > 2M, \end{cases}$$

with $0 \leq f^M(v) \leq v^2$. Here, the constant $C = C(M)$ is chosen to ensure the following property,

$$(3.1) \quad |(f^M)'|_\infty + |g^M|_\infty \leq C(M), \quad |(g^M)'|_\infty \leq 1.$$

The functions f^M, g^M are simply appropriate smooth truncations of the functions v^2 and v appearing in (1.1b). We define also $F^M(v) = \int_0^v f^M(s) ds$.

Now, we consider the auxiliary Cauchy problem,

$$(3.2a) \quad i\partial_t u + \partial_{xx} u = \beta|u|^2 u + \alpha g^M(v)u$$

$$(3.2b) \quad \partial_t v + \partial_x^3 v + \partial_x f^M(v) = \gamma \partial_x ((g^M)'(v)|u|^2)$$

$$(3.2c) \quad u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad x \in (0, L), \quad L > 0,$$

$$(3.2d) \quad u(0, t) = u(L, t) = 0, \quad v(0, t) = v(L, t) = 0, \quad t \in [0, T], \quad T > 0.$$

and we propose the following semidiscrete finite difference approximation of (3.2):

$$(3.3a) \quad i\partial_t u^h + \Delta^h u^h = \beta|u^h|^2 u^h + \alpha g^M(v^h)u^h$$

$$(3.3b) \quad \partial_t v^h + D^3 v^h + D_0 f^M(v^h) = \gamma D_0 ((g^M)'(v^h)|u^h|^2)$$

$$(3.3c) \quad u^h(0) = u_0^h, \quad v^h(0) = v_0^h,$$

$$(3.3d) \quad u^h(t), v^h(t) \in X_J, \quad t \in [0, T].$$

We will need some conservation laws of the auxiliary system (3.2). A first result is as follows:

Proposition 3.1. *Let $I \subset \mathbb{R}$ be an interval and let $(u, v) \in (L_{\text{loc}}^\infty(I; H_0^1(0, L)))^2$ be a solution of the auxiliary system (3.2). Then, for each $t, s \in I$, we have*

$$(3.4) \quad \mathcal{M}^M(t) := \|u(t)\|_2 = \|u(s)\|_2 = \mathcal{M}^M(s)$$

$$(3.5) \quad \mathcal{Q}^M(t) := \alpha \int_0^L v^2(x, t) dx + 2\gamma \operatorname{Im} \int_0^L u(x, t) \partial_x \bar{u}(x, t) dx = \mathcal{Q}^M(s).$$

Proof. Although the estimates (3.4), (3.5) are formally easy to obtain, the rigorous justification of (3.5) is nontrivial and requires techniques from semigroup theory.

First, since $u(t) \in H_0^1(0, L)$, from the equation (3.2a) we deduce that

$$\operatorname{Im} \langle i\partial_t u, \bar{u} \rangle_{H^{-1} \times H_0^1} + \operatorname{Im} \langle \partial_{xx} u, \bar{u} \rangle_{H^{-1} \times H_0^1} = 0,$$

and so,

$$\frac{d}{dt} \int_0^L |u|^2(t) dx = 0,$$

which gives (3.4).

For the proof of (3.5), we follow the ideas of Kato [7, Lemma 3.1]. We point out that $-\partial_x^3$ is a skew-adjoint operator on $L^2(0, L)$ with domain $H^3(0, L) \cap H_0^2(0, L)$ and it generates a group of isometries $U_K(t)$ on $L^2(0, L)$. Also, $i\Delta$ is a skew-adjoint operator on $L^2(0, L)$ with domain $H^2(0, L) \cap H_0^1(0, L)$ and so it generates a group of isometries $U_S(t)$ on $L^2(0, L)$. Now, we write the equations (3.2) in integral form,

$$(3.6) \quad u(t) = U_S(t-s)u(s) + \int_s^t U_S(t-r)m(r) dr,$$

$$(3.7) \quad v(t) = U_K(t-s)v(s) + \int_s^t U_K(t-r)n(r) dr,$$

with

$$\begin{aligned} m(r) &= -i\alpha g^M(v(r))u(r) - i\beta|u(r)|^2u(r), \\ n(r) &= -\partial_x f^M(v(r)) + \gamma\partial_x((g^M)'(v(r))|u(r)|^2). \end{aligned}$$

The formulas (3.6), (3.7) are verified in $L^2(0, L)$ and, since $\partial_x m \in L_{\text{loc}}^\infty(I; L^2(0, L))$, we have also

$$(3.8) \quad \partial_x u(t) = U_S(t-s)\partial_x u(s) + \int_s^t U_S(t-r)\partial_x m(r) dr$$

in $L^2(0, L)$.

Using the isometric property of $U_K(t)$ and $U_S(t)$, we easily deduce from (3.6)–(3.8)

$$(v(t), v(t)) = (v(s), v(s)) + 2 \int_s^t (v(r), n(r)) dr,$$

$$\text{Im}(u(t), \partial_x u(t)) = \text{Im}(u(s), \partial_x u(s)) + 2 \int_s^t \text{Im}(m(r), \partial_x u(r)) dr.$$

Since $u(r), v(r) \in H_0^1(0, L)$, we obtain from the expressions of $m(r)$ and $n(r)$

$$\begin{aligned} \text{Im}(m(r), \partial_x u(r)) &= -\text{Re} \alpha \int_0^L g^M(v(r))u(r)\partial_x \bar{u}(r) dx \\ &\quad - \beta \text{Re} \int_0^L |u(r)|^2 u(r)\partial_x \bar{u}(r) dx \\ &= \frac{\alpha}{2} \int_0^L (g^M)'(v(r))\partial_x v(r)|u(r)|^2 dx. \end{aligned}$$

It follows that

$$2\gamma \text{Im}(m(r), \partial_x u(r)) = -\alpha(v(r), n(r)),$$

which implies the conclusion (3.5). This completes the proof of Proposition 3.1. \square

The following result establishes the convergence of the approximations (3.3) to a global solution of the truncated problem (3.2), and the crucial energy estimate (3.12).

Proposition 3.2. *Let α, β, γ be such that $\alpha\gamma > 0$. For each $M > 0$ let $(u^{h,M}, v^{h,M}) \in (C([0, T]; X_J))^2$ be the solution of (3.3) with initial data (u_0^h, v_0^h) such that $\mathbf{P}_1^h u_0^h \rightarrow u_0$, $\mathbf{P}_1^h v_0^h \rightarrow v_0$ in $H^1(0, L)$ as $h \rightarrow 0$. Then, up to a subsequence,*

$$(3.9) \quad \mathbf{P}_1^h u^{h,M} \xrightarrow{*} u^M, \quad \mathbf{P}_1^h v^{h,M} \xrightarrow{*} v^M \quad \text{in } L^\infty([0, T]; H^1(0, L)),$$

$$(3.10) \quad \mathbf{P}_1^h u^{h,M} \rightarrow u^M, \quad \mathbf{P}_1^h v^{h,M} \rightarrow v^M \quad \text{in } L^\infty([0, T]; L^2(0, L)),$$

with

$$(3.11) \quad (u^M, v^M) \in (L^\infty([0, T]; H_0^1(0, L)))^2 \cap (C([0, T]; H_0^1(0, L)) \times C([0, T]; L^2(0, L))), \quad T > 0,$$

a global weak solution of the truncated system (3.2). Moreover, the following energy estimate is valid,

$$(3.12) \quad \begin{aligned} \mathcal{E}^M(t) &:= \int_0^L \left\{ \gamma |\partial_x u^M(t)|^2 + \frac{\alpha}{2} |\partial_x v^M(t)|^2 + \alpha \gamma g^M(v^M(t)) |u^M(t)|^2 \right. \\ &\quad \left. - \alpha F^M(v^M(t)) + \frac{\beta \gamma}{2} |u^M(t)|^4 \right\} dx \leq \mathcal{E}^M(0), \end{aligned}$$

for all $t \in [0, T]$.

We postpone the proof of Proposition 3.2 until Section 4, and proceed to prove Theorem 2.1. The goal is to prove an L^∞ bound for u^M, v^M independent of the truncation parameter M , using the energy inequality (3.12). Once this is achieved, it is clear from the definition of f^M and g^M that taking M large enough yields a solution of the original problem, (1.1).

Let us define $\mathcal{M}_0 = \mathcal{M}(0)$, $\mathcal{Q}_0 = \mathcal{Q}(0)$ (see (3.4),(3.5)), and set

$$(3.13) \quad \mathcal{E}_0 = |\gamma| \|\partial_x u_0\|_2^2 + \frac{|\alpha|}{2} \|\partial_x v_0\|_2^2 + |\alpha\gamma| \|v_0\|_2 \|u_0\|_4^2 + \frac{|\alpha|}{3} \|v_0\|_3^3 + \frac{|\beta\gamma|}{2} \|u_0\|_4^4.$$

Observe that $|\mathcal{E}^M(0)| \leq \mathcal{E}_0$ for all $M > 0$ (see (3.12)).

For simplicity, in what follows we omit the superscript M from the solutions (u^M, v^M) of the system (3.2) obtained in Proposition 3.2.

First, note that the energy inequality (3.12) gives

$$(3.14) \quad \begin{aligned} |\gamma| \|\partial_x u(t)\|_2^2 + \frac{|\alpha|}{2} \|\partial_x v(t)\|_2^2 \\ \leq (\mathcal{E}_0 + |\alpha\gamma| \|v(t)\|_2 \|u(t)\|_4^2 + \frac{|\alpha|}{3} \|v(t)\|_3^3 + \frac{|\beta\gamma|}{2} \|u(t)\|_4^4). \end{aligned}$$

Next, from (3.5) we have

$$(3.15) \quad \|v(t)\|_2^2 \leq \frac{1}{|\alpha|} (|\mathcal{Q}_0| + 2|\gamma| \|u_0\|_2 \|\partial_x u(t)\|_2).$$

Let now $m = \min\{|\gamma|, |\alpha|/2\}$. Using again Gagliardo–Nirenberg and Young inequalities, we deduce from (3.14)–(3.15) (as in [4])

$$\begin{aligned} \|\partial_x u(t)\|_2^2 + \|\partial_x v(t)\|_2^2 &\leq \frac{1}{m} (|\gamma| \|\partial_x u(t)\|_2^2 + \frac{|\alpha|}{2} \|\partial_x v(t)\|_2^2) \\ &\leq \frac{1}{m} (\mathcal{E}_0 + |\alpha\gamma| \|v(t)\|_2 \|u(t)\|_4^2 + \frac{|\alpha|}{3} \|v(t)\|_3^3 + \frac{|\beta\gamma|}{2} \|u(t)\|_4^4) \\ &\leq C (\mathcal{E}_0 + \|v(t)\|_2^2 + \|v(t)\|_3^3 + \|u(t)\|_4^4) \\ &\leq C (\mathcal{E}_0 + |\mathcal{Q}_0| + |\mathcal{Q}_0|^{5/3} + \mathcal{M}_0 + \mathcal{M}_0^3 + \mathcal{M}_0^5) \\ &\quad + \frac{1}{2} (\|\partial_x u(t)\|_2^2 + \|\partial_x v(t)\|_2^2), \end{aligned}$$

with $C = C(\alpha, \beta, \gamma)$ only depending on the parameters α, β, γ . Therefore

$$\|\partial_x u(t)\|_2^2 + \|\partial_x v(t)\|_2^2 \leq 2C (\mathcal{E}_0 + |\mathcal{Q}_0| + |\mathcal{Q}_0|^{5/3} + \mathcal{M}_0 + \mathcal{M}_0^3 + \mathcal{M}_0^5) := K_0.$$

Note that K_0 is independent of M . Finally, since from (3.15),

$$\begin{aligned} \|v(t)\|_\infty^2 &\leq 2\|v(t)\|_2 \|\partial_x v(t)\|_2 \leq \|v(t)\|_2^2 + \|\partial_x v(t)\|_2^2 \\ &\leq \frac{1}{|\alpha|} |\mathcal{Q}_0| + \left| \frac{\gamma}{\alpha} \right| \left(\|u_0\|_2^2 + \|\partial_x u(t)\|_2^2 \right) + \|\partial_x v(t)\|_2^2, \end{aligned}$$

we obtain

$$(3.16) \quad \|v(t)\|_\infty \leq \left(\frac{1}{|\alpha|} |\mathcal{Q}_0| + \left| \frac{\gamma}{\alpha} \right| \mathcal{M}_0 + (1 + |\gamma/\alpha|) K_0 \right)^{1/2} := \bar{K},$$

with \bar{K} independent of M but depending on the initial data. Therefore, if the truncation parameter M in (3.2) satisfies $M > \bar{K}$, we conclude by (3.16) and the definition of f^M, g^M that $(u, v) := (u^M, v^M)$ is actually a solution of the Schrödinger–KdV system (1.1). This completes the proof of Theorem 2.1.

4. PROOF OF PROPOSITION 3.2

First of all, we need the following lemma concerning the global existence of the solution of the discrete problem (3.3). Due to the fact that the problem (3.3) is truncated, we are also able to obtain the essential H^1 estimate (4.1). For simplicity, we will omit the superscript M .

Lemma 4.1. *Let $\alpha, \beta, \gamma \in \mathbb{R}$ be such that $\alpha\gamma > 0$. Fix $J \in \mathbb{N}$, $L > 0$, $h = L/(J+1)$, and $(u_0^h, v_0^h) \in X_{J,\mathbb{C}} \times X_{J,\mathbb{R}}$. Then, for each $T > 0$, there exists a unique solution*

$$(u^h, v^h) \in C([0, T]; X_{J,\mathbb{C}}) \times C([0, T]; X_{J,\mathbb{R}})$$

of the problem (3.3). Moreover, the following estimate is valid,

$$(4.1) \quad \begin{aligned} & \|D_+ u^h(t)\|_2^2 + \|D_+ v^h(t)\|_2^2 + \|u^h(t)\|_2^2 + \|v^h(t)\|_2^2 \\ & \leq C(\|u_0^h\|_2, \|v_0^h\|_2, \|D_+ u_0^h\|_2, \|D_+ v_0^h\|_2, T, M), \end{aligned}$$

for all $t \in [0, T]$.

Proof. Let $S_h(t) = e^{i\Delta^h t}$, $G_h(t) = e^{-D^3 t}$ be the unitary groups generated by the discrete operators $i\Delta^h$ and $-D^3$ in the X_J space. The problem (3.3) can be written in the usual semigroup framework, as an integral equation in the $X_{J,\mathbb{C}} \times X_{J,\mathbb{R}}$ space:

$$(4.2) \quad \begin{aligned} u^h(t) &= S_h(t)u_0^h + \int_0^t S_h(t-s)J_S(u^h(s), v^h(s)) ds =: \Phi_1(u^h, v^h), \\ v^h(t) &= G_h(t)v_0^h + \int_0^t G_h(t-s)J_K(u^h(s), v^h(s)) ds =: \Phi_2(u^h, v^h), \end{aligned}$$

where

$$\begin{aligned} J_S(u^h, v^h) &= -i\alpha g(v^h)u^h - i\beta|u^h|^2 u^h, \\ J_K(u^h, v^h) &= -D_0 f(v^h) + \gamma D_0(g'(v^h)|u^h|^2). \end{aligned}$$

For $R > \|u_0^h\|_2 + \|v_0^h\|_2$ and $T > 0$ we consider the product space $B_{R,\mathbb{C}}^T \times B_{R,\mathbb{R}}^T$, with

$$B_{R,\mathbb{C}}^T = \{w \in C([0, T]; X_{J,\mathbb{C}}) : \|w\|_{L^\infty([0, T]; X_J)} \leq R\}$$

and similarly for $B_{R,\mathbb{R}}^T$.

By (1.7), it is now a simple matter to prove that there exists $T > 0$ such that the map

$$(u^h, v^h) \in B_{R,\mathbb{C}}^T \times B_{R,\mathbb{R}}^T \mapsto \Phi(u^h, v^h) := (\Phi_1(u^h, v^h), \Phi_2(u^h, v^h))$$

is a strict contraction on the complete metric space $B_{R,\mathbb{C}}^T \times B_{R,\mathbb{R}}^T$. Thus, by the Banach fixed-point theorem, we obtain a unique local in time solution (u^h, v^h) of the problem (3.3) in the space $C([0, T]; X_{J,\mathbb{C}}) \times C([0, T]; X_{J,\mathbb{R}})$.

To obtain a global solution, we must estimate the l^2 -norm of $u^h(t)$ and $v^h(t)$, for each fixed h . From the equation (3.3a), the conservation of the l^2 -norm of $u^h(t)$ follows easily by taking the scalar product with u^h and summation by parts. Applying the same procedure to equation (3.3b), we find

$$(4.3) \quad \frac{1}{2} \partial_t \|v^h(t)\|_2^2 = \sum_{j=1}^J h f(v_j) D_0 v_j - \sum_{j=1}^J h g'(v_j) |u_j|^2 D_0 v_j.$$

From the definition of f, g , the conservation of the l^2 -norm of u^h and the fact that, for h fixed, $\|\cdot\|_\infty \leq C(h)\|\cdot\|_2$, we derive that

$$\|v^h(t)\|_2^2 \leq \|v_0^h\|_2^2 + C(h) \int_0^t \|v^h(s)\|_2^2 ds.$$

The conclusion now follows from a Gronwall argument.

It remains to prove the inequality (4.1). In addition to the conservation of the l^2 -norm of u^h , we have the conservation of the discrete energy:

$$(4.4) \quad \begin{aligned} E^h(t) &:= \gamma \|D_+ u^h(t)\|_2^2 + \frac{\alpha}{2} \|D_+ v^h(t)\|_2^2 + \frac{\beta\gamma}{2} \|u^h(t)\|_4^4 \\ &+ \alpha\gamma \sum_{j=1}^J hg(v_j)|u_j|^2 - \alpha \sum_{j=1}^J hF(v_j) = E^h(0). \end{aligned}$$

To prove this identity, we proceed in the same way as in the continuous case [12]: Take the scalar product in X_J of the equation (3.3a) with $\partial_t \bar{u}^h$, take the real part, and use the equation (3.3b) and the skew-adjoint properties of the operators D_0 and D^3 .

Now we return to (4.3) and observe that from $f(v_j) = f(v_j) - f(0) = f'(\theta_j)v_j$, $|g'| \leq 1$ and (1.6) we find

$$\begin{aligned} \partial_t \|v^h\|_2^2 &\leq C(M)(\|D_+ v^h\|_2^2 + \|v^h\|_2^2) + C\|u^h\|_\infty \|D_+ v^h\|_2 \\ &\leq C(M)(\|D_+ v^h\|_2^2 + \|D_+ u^h\|_2^2 + \|v^h\|_2^2). \end{aligned}$$

Integrating on $(0, t)$ and using Gronwall's lemma, we obtain

$$(4.5) \quad \|v^h(t)\|_2 \leq a_1(t, M) + a_2(t, M) \int_0^t \|D_+ v^h(s)\|_2^2 + \|D_+ u^h(s)\|_2^2 ds$$

for some continuous functions a_1, a_2 . On the other hand, from the conservation of the energy (4.4) and using (1.5) we get

$$(4.6) \quad \begin{aligned} \|D_+ u^h(s)\|_2^2 + \|D_+ v^h(s)\|_2^2 &\leq C(u_0^h, v_0^h) + C_1 \|D_+ u^h(s)\|_2 \\ &+ C_2 \sum_{j=1}^J hg(v_j)|u_j|^2 + C_3 \sum_{j=1}^J hF(v_j). \end{aligned}$$

But now, the definition of f^M allows us (roughly) to bound $F^M(v)$ by $C(M)v^2$. This is essential in view of the desired H^1 estimate (4.1), since these terms would otherwise be cubic. We have

$$\begin{aligned} \sum_{j=1}^J h|F(v_j)| &= \sum_{j=1}^J h|F(v_j) - F(0)| = \sum_{j=1}^J h|f(\theta_j)v_j| \\ &\leq \sum_{j=1}^J hf(\theta_j)^2 + \|v^h\|_2^2, \end{aligned}$$

for some θ_j between 0 and v_j . Now,

$$\sum_{j=1}^J hf(\theta_j)^2 = \sum_{|v_j| \leq M^2+1} hf(\theta_j)^2 + \sum_{|v_j| > M^2+1} hf(\theta_j)^2.$$

Recall the definition of the truncated functions in (3.1). For the first sum, we have $f(\theta_j)^2 \leq v_j^4 \leq (M^2 + 1)^2 v_j^2$, and for the second sum we have $f(\theta_j)^2 \leq v_j^2$. Thus we obtain

$$\sum_{j=1}^J h|F(v_j)| \leq C(M)\|v^h\|_2^2.$$

Similarly, since the l^2 -norm of u^h is conserved, we find

$$\sum_{j=1}^J h|g(v_j)||u_j|^2 \leq \|g\|_\infty \|u^h\|_2^2 \leq C(M)\|u_0^h\|_2^2.$$

These estimates together with (4.5) and (4.6) give us

$$\begin{aligned} \|D_+ u^h(t)\|_2^2 + \|D_+ v^h(t)\|_2^2 &\leq C(u_0^h, v_0^h, M) + C(M)\|v^h(t)\|_2^2 \\ &\leq C(u_0^h, v_0^h, M) + C(t, M) \int_0^t \|D_+ u^h(s)\|_2^2 + \|D_+ v^h(s)\|_2^2 ds, \end{aligned}$$

where $C(u_0^h, v_0^h, M) = C(\|u_0^h\|_2, \|D_+ u_0^h\|_2, \|v_0^h\|_2, \|D_+ v_0^h\|_2, M)$. A Gronwall argument, (4.5), and the conservation of $\|u^h\|_2$ give the conclusion (4.1). This completes the proof of Lemma 4.1. \square

Proof of Proposition 3.2. We will use the interpolators $\mathbf{P}_1^h, \mathbf{P}_0^h$ defined in (1.2). Since $\partial_x \mathbf{P}_1^h = \mathbf{P}_0^h D_+$, it follows from the hypothesis $\mathbf{P}_1^h u_0^h \rightarrow u_0$, $\mathbf{P}_1^h v_0^h \rightarrow v_0$ in $H_0^1(0, L)$, (1.4) and (4.1) that

$$\|\mathbf{P}_1^h u^h(t)\|_{H^1(0, L)} \leq C, \quad \|\mathbf{P}_1^h v^h(t)\|_{H^1(0, L)} \leq C, \quad t \in [0, T]$$

with $C = C(\|u_0\|_{H^1}, \|v_0\|_{H^1}, T, M)$. Thus, using the compactness of the embedding of $H^1(0, L)$ into $L^2(0, L)$, we obtain, as $h \rightarrow 0$ (for a subsequence),

$$(4.7) \quad \begin{aligned} \mathbf{P}_1^h u^h &\overset{*}{\rightharpoonup} u, & \mathbf{P}_1^h v^h &\overset{*}{\rightharpoonup} v & \text{in } L^\infty([0, T]; H^1(0, L)), \\ \mathbf{P}_1^h u^h &\rightarrow u, & \mathbf{P}_1^h v^h &\rightarrow v & \text{in } L^\infty([0, T]; L^2(0, L)), \end{aligned}$$

for some $u, v \in H_0^1(0, L)$. Also, we have from (1.3),

$$(4.8) \quad \mathbf{P}_0^h u^h \rightarrow u, \quad \mathbf{P}_0^h v^h \rightarrow v \text{ in } L^\infty([0, T]; L^2(0, L)).$$

To prove that u, v are solutions to the system (3.2), we apply the piecewise constant interpolator \mathbf{P}_0^h to the equations (3.3a), (3.3b):

$$(4.9) \quad i\partial_t \mathbf{P}_0^h u^h + \mathbf{P}_0^h \Delta^h u^h = \beta \mathbf{P}_0^h (|u^h|^2 u^h) + \alpha \mathbf{P}_0^h (g^M(v^h) u^h)$$

$$(4.10) \quad \partial_t \mathbf{P}_0^h v^h + \mathbf{P}_0^h D^3 v^h + \mathbf{P}_0^h D_0 f^M(v^h) = \gamma \mathbf{P}_0^h D_0 ((g^M)'(v^h) |u^h|^2).$$

From (1.3), (4.1) we have

$$(4.11) \quad \mathbf{P}_1^h f(v^h) - \mathbf{P}_0^h f(v^h) \rightarrow 0 \text{ in } L^\infty([0, T]; L^2(0, L))$$

and, since the piecewise constant interpolator commutes with nonlinearities, it follows from (4.8) that

$$(4.12) \quad \mathbf{P}_0^h f(v^h) = f(\mathbf{P}_0^h v^h) \rightarrow f(v) \text{ in } L^\infty([0, T]; L^2(0, L)).$$

On the other hand, using (4.1),

$$\|\partial_x \mathbf{P}_1^h f(v^h)\|_2 = \|\mathbf{P}_0^h D_+ f(v^h)\|_2 = \|D_+ f(v^h)\|_2 \leq C(M),$$

and, from (4.11),(4.12) we deduce

$$\mathbf{P}_1^h f(v^h) \xrightarrow{*} f(v) \text{ in } L^\infty([0, T]; H^1(0, L))$$

and so,

$$\partial_x \mathbf{P}_1^h f(v^h) \xrightarrow{*} \partial_x f(v) \text{ in } L^\infty([0, T]; L^2(0, L)).$$

Similarly, and using also (4.8), we find

$$\mathbf{P}_0^h(|u^h|^2 u^h) = |\mathbf{P}_0^h u^h|^2 \mathbf{P}_0^h u^h \rightarrow |u|^2 u \text{ in } L^\infty([0, T]; L^2(0, L)),$$

$$\mathbf{P}_0^h(g(v^h)u^h) = g(\mathbf{P}_0^h v^h) \mathbf{P}_0^h u^h \rightarrow g(v)u \text{ in } L^\infty([0, T]; L^2(0, L)),$$

$$\mathbf{P}_0^h D_0(g'(v^h)|u^h|^2) = \partial_x \mathbf{P}_1^h(g'(v^h)|u^h|^2) \xrightarrow{*} \partial_x(g'(v)|u|^2) \text{ in } L^\infty([0, T]; L^2(0, L)),$$

which allows us to pass to the limit on the corresponding terms in the weak formulation of the equations (1.1).

It remains to analyze the terms $\mathbf{P}_0^h \Delta^h u^h$ and $\mathbf{P}_0^h D^3 v^h$. Let $\phi \in \mathcal{D}(0, L)$ be a test function. We have

$$\begin{aligned} \langle \mathbf{P}_0^h D^3 v^h, \phi \rangle &= \sum_{j=2}^{J-1} \int_{x_j}^{x_{j+1}} \mathbf{P}_0^h D^3 v^h \phi \, dx = \sum_{j=2}^{J-1} D_0 D_- D_+ v_j \int_{x_j}^{x_{j+1}} \phi(x) \, dx \\ &= \sum_{j=2}^{J-1} D_+ v_j \int_{x_j}^{x_{j+1}} \frac{1}{2h^2} (\phi(x-2h) - \phi(x-h) + \phi(x) - \phi(x+h)) \, dx \end{aligned}$$

and so, by Taylor expansion of ϕ ,

$$|\langle \mathbf{P}_0^h D^3 v^h, \phi \rangle| \leq C \sum_{j=2}^{J-1} h |D_+ v_j| \|\phi''\|_\infty \leq C \left(\sum_{j=2}^{J-1} h |D_+ v_j|^2 \right)^{1/2} \|\phi\|_{H^3(0, L)}.$$

Hence, from (4.1) we obtain

$$(4.13) \quad \|\mathbf{P}_0^h D^3 v^h\|_{L^\infty([0, T]; H^{-3}(0, L))} \leq C.$$

If we now take a test function $\varphi \in \mathcal{D}((0, T) \times (0, L))$, we may compute in the sense of distributions

$$\begin{aligned} \langle \mathbf{P}_0^h D^3 v^h, \varphi \rangle &= \int_0^T \sum_{j=1}^J D^3 v_j \int_{x_j}^{x_{j+1}} \varphi(x, t) \, dx \, dt \\ &= - \int_0^T \sum_{j=1}^J v_j \int_{x_j}^{x_{j+1}} (\partial_x^3 \varphi + \mathcal{O}(h)) \, dx \, dt \\ &= - \langle \mathbf{P}_0^h v^h, \partial_x^3 \varphi \rangle + \mathcal{O}(h) \rightarrow - \langle v, \partial_x^3 \varphi \rangle = \langle \partial_x^3 v, \varphi \rangle \end{aligned}$$

as $h \rightarrow 0$. Hence, we deduce from (4.13) that

$$\mathbf{P}_0^h D^3 v^h \xrightarrow{*} \partial_x^3 v \text{ in } L^\infty([0, T]; H^{-3}(0, L)).$$

In a similar way we prove that

$$\mathbf{P}_0^h \Delta^h u^h \xrightarrow{*} \Delta u \text{ in } L^\infty([0, T]; H^{-2}(0, L))$$

and using the equations,

$$i \partial_t \mathbf{P}_0^h u^h \xrightarrow{*} i \partial_t u \text{ in } L^\infty([0, T]; H^{-2}(0, L))$$

$$\partial_t \mathbf{P}_0^h v^h \xrightarrow{*} \partial_t v \text{ in } L^\infty([0, T]; H^{-3}(0, L)).$$

Therefore, taking the limit $h \rightarrow 0$ in the weak formulation of the equations (4.9),(4.10) we obtain a weak solution

$$(u, v) \in (L^\infty([0, T]; H_0^1(0, L)))^2 \\ \cap (C([0, T]; H^{-2}(0, L)) \times C([0, T]; H^{-3}(0, L))), \quad T > 0,$$

of the problem (3.2). To prove (3.11), recall that this solution satisfies the integral system (3.6),(3.7). Since

$$\|m(u, v)\|_{H^1} \leq C(\|u_0\|_{H^1}, \|v_0\|_{H^1}),$$

$$\|n(u, v)\|_{H^1} \leq C(\|u_0\|_{H^1}, \|v_0\|_{H^1}),$$

we deduce from (4.13) that

$$(u, v) \in C([0, T]; H^1) \times C([0, T]; L^2).$$

It remains to prove the energy inequality (3.12). Let us write the discrete energy (4.4) in the form

$$(4.14) \quad E^h(t) := \gamma \|\partial_x \mathbf{P}_1^h u^h(t)\|_2^2 + \frac{\alpha}{2} \|\partial_x \mathbf{P}_1^h v^h(t)\|_2^2 + \frac{\beta\gamma}{2} \|\mathbf{P}_0^h u^h(t)\|_4^4 \\ + \alpha\gamma \int_0^L g(\mathbf{P}_0^h v^h) |\mathbf{P}_0^h u^h|^2 dx - \alpha \int_0^L F(\mathbf{P}_0^h v^h) dx = E^h(0).$$

Now we recall the weak and strong convergences (4.7),(4.8). From the last term on the left-hand side of (4.14), and since $|F'(\xi)| \leq C|\xi|^2$, we find

$$\int_0^L |F(\mathbf{P}_0^h v^h(t)) - F(v(t))| dx \leq C \|\mathbf{P}_0^h v^h(t) - v(t)\|_2 \|(\mathbf{P}_0^h v^h(t))^2 + v^2(t)\|_2 \\ \leq C \|\mathbf{P}_0^h v^h(t) - v(t)\|_2 \rightarrow 0 \quad (h \rightarrow 0)$$

and so

$$\int_0^L F(\mathbf{P}_0^h v^h(t)) dx \rightarrow \int_0^L F(v(t)) dx.$$

Note that here it is essential that the spatial domain $(0, L)$ be bounded. Indeed, a version of the energy inequality (3.12) on the whole line cannot be obtained using the available convergences (4.7),(4.8), which are local in space.

Finally, from the strong convergence in $H^1(0, L)$ of the initial data (u_0^h, v_0^h) and using the lower semi-continuity of the H^1 norm, we easily obtain from (4.14), in the limit $h \rightarrow 0$, the conclusion (3.12): $\mathcal{E}(t) \leq \mathcal{E}(0)$, $t \in [0, T]$. This completes the proof of Proposition 3.2. \square

5. NUMERICAL EXPERIMENTS

In this section, we present some numerical computations using a fully discrete version of the method (2.1). We emphasize that these simulations are for the sake of illustration of our convergence result only. In particular, it would be interesting to perform more extensive numerical tests, such as determining the order of convergence, or employing more sophisticated time discretizations, which we do not perform here.

Given some time step $\tau > 0$, a spatial mesh size h , and initial data $(u_{0j}, v_{0j})_{j=0,\dots,J+1}$, we consider for $n \geq 0$ the following algorithm. Set $u_j^{n+1/2} = \frac{1}{2}(u_j^{n+1} + u_j^n)$ and solve

$$(5.1) \quad \begin{aligned} i\frac{1}{\tau}(u_j^{n+1} - u_j^n) + \Delta^h u_j^{n+1/2} &= |u_j^{n+1/2}|^2 u_j^{n+1/2} + v_j^n u_j^{n+1/2}, \\ \frac{1}{\tau}(v_j^{n+1} - v_j^n) + D^3 v_j^{n+1} + D_0(v_j^{n+1})^2 &= D_0(|u_j^n|^2), \end{aligned}$$

for $j = 0, \dots, J+1$. This corresponds to a semi-implicit Crank–Nicolson scheme for the Schrödinger equation and a fully implicit Euler scheme for the KdV equation. Because of the nonlinear terms, we perform a Newton iteration at each time step with a tolerance of 10^{-6} . At each Newton iteration we solve independently a tridiagonal system for the first equation of (5.1) by a standard direct method, and the pentadiagonal system issuing from the second equation by an LU decomposition method.

5.1. Comparison with exact solutions. We now test our scheme and illustrate our convergence result. We will simulate the following system,

$$(5.2) \quad \begin{cases} i\partial_t u + \partial_{xx} u = \alpha v u - |u|^2 u \\ \partial_t v + \partial_x^3 v + v\partial_x v = \frac{\alpha}{2}\partial_x |u|^2, \end{cases}$$

which is the same as (1.1) for a special choice of the parameters, except for the quasilinear term in the KdV equation which is formally equivalent to $\frac{1}{2}\partial_x(v^2)$. In [6] we can find the following exact traveling wave solutions to (5.2),

$$(u, v) = (e^{i\omega t} e^{ixc/2} \phi(x - ct), \psi(x - ct)),$$

with

$$\phi(y) = \frac{\sqrt{2c^*(1+6\alpha)}}{\cosh(\sqrt{c^*}y)}, \quad \psi(y) = \frac{12c^*}{\cosh^2(\sqrt{c^*}y)}.$$

Here, $\alpha \in [-1/6, 0]$ and $\omega \in \mathbb{R}$ are given, and $2c = 1 + \sqrt{1 + \frac{\alpha}{3}(1+6\alpha)}$, $c^* = c^2/4 + \omega^2$. We chose $\alpha = -1/12$ and $\omega = 0$ for the simulations below. This gives a traveling wave speed $c = 0.996516$.

In Figure 1 we present (in logarithmic scale) the relative L^2 error computed at $T = 5$ for a time step $\tau = 0.0001$ as a function of the mesh size. The computational domain is $[-20, 50]$ and the number of spatial points ranges from 500 to 2500. In Figure 2 we present the relative L^2 error for a similar computation with $T = 30$ and a time step $\tau = 0.0005$. The number of spatial points ranges from 200 to 600.

It can be seen that the exact solution is approximated rather well by our simple numerical scheme, especially bearing in mind that the simulations (which, again, serve only to illustrate our results) were performed in a few minutes on a laptop running at 2.4 GHz.

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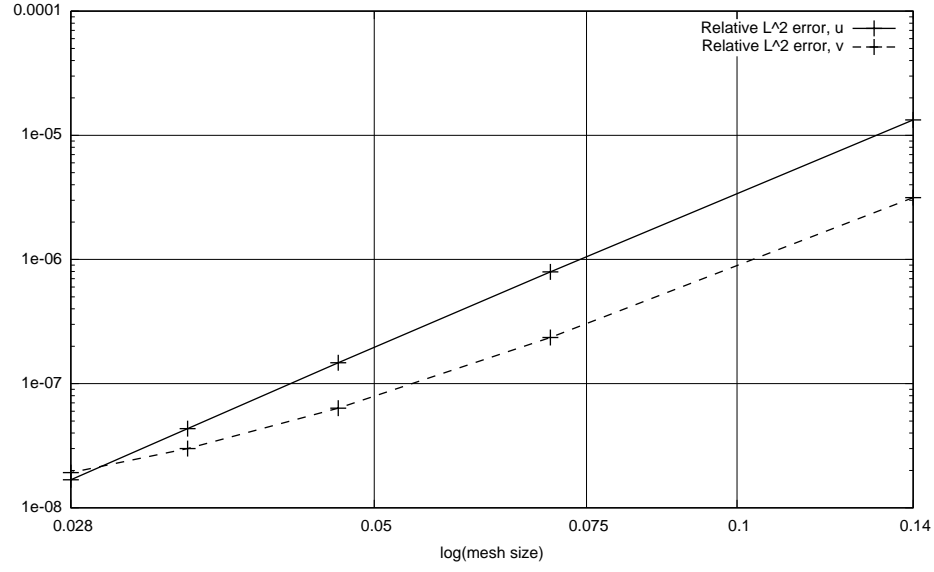


FIGURE 1. Relative L^2 error, $T = 5$, $\tau = 0.0001$, as a function of mesh size. 500 to 2500 spatial points.

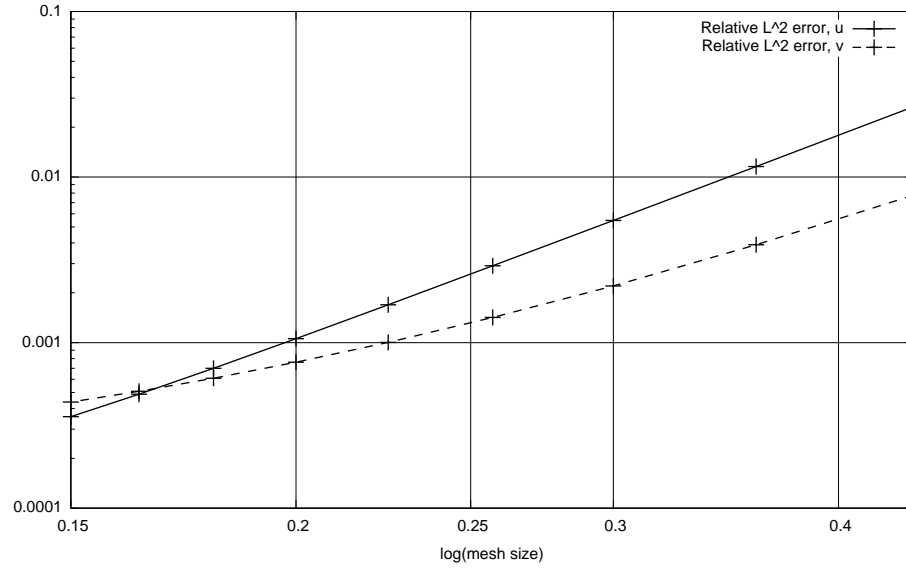


FIGURE 2. Relative L^2 error, $T = 30$, $\tau = 0.0005$, as a function of mesh size. 200 to 600 spatial points.

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